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Transient conduction in a three-dimensional composite slab

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INTRODUCTION

IN A RECENT issue of this journal, Salt [1, 2] examined in two consecutive papers the eigenvalues and eigenfunctions of the eigenvalue problem associated with the solution of transient heat conduction in a two-dimensional composite slab having three of its boundaries insulated with the fourth boundary parallel to the layers subjected to a uniform temperature. By considering the physical significance of the eigenvalues it was concluded that:

> while it is possible to have a temperature variation across a fully insulated composite slab with no temperature variation along it, it is impossible to have temperature variation along the slab without having temperature variation across it.

In the present brief note we analyze the three-dimensional version of the problem considered by Salt [1, 2] and show that similar results and conclusions are readily obtainable as a very special case of the general solutions given in refs. [3, 4]. Furthermore, the recently developed sign-count method [4–6] is applicable for the solution of the two- or three-dimensional eigenvalue problems associated with heat conduction in multilayer slabs.

STATEMENT OF THE PROBLEM

We consider three-dimensional transient conduction in a nonhomogeneous finite medium in which the thermal and physical properties vary only in the z-direction. The mathematical formulation of the problem is taken as

$$w(z)\frac{\partial T(x,y,z)}{\partial t} = k(z)\frac{\partial^2 T}{\partial x^2} + k(z)\frac{\partial^2 T}{\partial y^2} + \frac{\partial}{\partial z}\left\{k(z)\frac{\partial T}{\partial z}\right\}, \quad \text{in} \quad 0 < x < a,$$
$$0 < y < b, \quad 0 < z < c, \quad \text{for} \quad t > 0 \quad (1a)$$

subject to the boundary conditions

$$\frac{\partial T(0,y,z,t)}{\partial x} = 0, \quad \frac{\partial T(a,y,z,t)}{\partial x} = 0 \quad (1b, c)$$

$$\frac{\partial T(x,0,z,t)}{\partial y} = 0, \quad \frac{\partial T(x,b,z,t)}{\partial y} = 0 \quad (1d, e)$$

$$\frac{\partial T(x,y,0,t)}{\partial z} = 0, \quad T(x,y,c,t) = 0 \quad (1f, g)$$

and the initial condition

$$T(x,y,z,0) = f(x,y,z), \quad (1h)$$

where $w(z) = c(z)\rho(z)$, the specific heat $c(z)$, the density $\rho(z)$ and the thermal conductivity $k(z)$ are known function of the z coordinate.

Clearly, the problem (1) describes as a special case a multilayer composite slab when $w(z)$ and $k(z)$ are chosen as stepwise functions in the z-direction, that is

$$k(z) = k_k, \quad w(z) = w_k \quad \text{for}$$
$$z_{k-1} < z < z_k, \quad k = 1, 2, \ldots, n. \quad (2)$$

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NOMENCLATURE

a, b, c	width, length and thickness of the slab
$f(x, y, z)$	initial temperature distribution
$k(z)$	known function of the z coordinate
l, m, n	integers
t	time
$T(x, y, z, t)$	temperature in the slab
x, y	positions along the slab
z	position across the slab
X, Y, Z	eigenfunctions defined by equations (5), (6) and (7)
$w(z)$	known function of the z coordinate.

Greek symbols

λ_l, ν_m	longitudinal eigenvalues
μ_{lmn}	eigenvalues
$\psi(x, y, z)$	eigenfunctions.

SOLUTION OF THE PROBLEM

The eigenvalue problem appropriate for the solution of the problem (1) is taken as

$$k(z)\frac{\partial^2\psi(x,y,z)}{\partial x^2}+k(z)\frac{\partial^2\psi}{\partial y^2}+\frac{\partial}{\partial z}\left\{k(z)\frac{\partial\psi}{\partial z}\right\}+\mu^2 w(z)\psi=0,\quad \text{in}\quad 0<x<a,$$

$$0<y<b,\quad 0<z<c \qquad (3a)$$

subject to the boundary conditions

$$\frac{\partial\psi(0,y,z)}{\partial x}=0,\quad \frac{\partial\psi(a,y,z)}{\partial x}=0 \qquad (3b,c)$$

$$\frac{\partial\psi(x,0,z)}{\partial y}=0,\quad \frac{\partial\psi(x,b,z)}{\partial y}=0 \qquad (3d,e)$$

$$\frac{\partial\psi(x,y,0)}{\partial z}=0,\quad \psi(x,y,c)=0. \qquad (3f,g)$$

To develop solution for this eigenvalue problem we assume a separation of variables in the form

$$\psi(x,y,z)=X(x)\cdot Y(y)\cdot Z(z) \qquad (4)$$

and split up the problem (3) into the following three, separated, one-dimensional eigenvalue problems

$$X''(x)+\lambda^2X(x)=0,\quad \text{in}\quad 0<x<a \qquad (5a)$$

$$X'(0)=0,\quad X'(a)=0 \qquad (5b,c)$$

$$Y''(y)+\nu^2Y(y)=0,\quad \text{in}\quad 0<y<b \qquad (6a)$$

$$Y'(0)=0,\quad Y'(b)=0 \qquad (6b,c)$$

and

$$\frac{d}{dz}\left\{k(z)\frac{dZ(z)}{dz}\right\}+\{\mu^2w(z)-(\lambda^2+\nu^2)k(z)\}Z(z)=0$$

$$\text{in}\quad 0<z<c \qquad (7a)$$

$$Z'(0)=0,\quad Z(c)=0 \qquad (7b,c)$$

The eigenvalues λ_l, ν_m ($l, m = 0, 1, 2,\ldots,\infty$), the eigenfunctions $X_l(x)$, $Y_m(y)$ and the normalization integrals N_l, N_m of the eigenvalue problems (5) and (6) are determined in ref. [4] as

$$\lambda_l=\frac{l\pi}{a},\quad X_l(x)=\cos(\lambda_l x),$$

$$N_l\equiv\int_0^a X_l^2(x)\,dx=\begin{cases}\frac{a}{2} & \text{for}\quad l=1,2,\ldots\\ a & \text{for}\quad l=0\end{cases} \qquad (8a,b,c)$$

$$\nu_m=\frac{m\pi}{b},\quad Y_m(y)=\cos(\nu_m y),$$

$$N_m\equiv\int_0^b Y_m^2(y)\,dy=\begin{cases}\frac{b}{2} & \text{for}\quad m=1,2,\ldots\\ b & \text{for}\quad m=0.\end{cases} \qquad (9a,b,c)$$

The eigenvalue problem (7) coincides with the classical Sturm–Liouville problem treated in ref. [5] when we set

$$(\lambda^2+\nu^2)k(z)\equiv d(z) \qquad (10)$$

where $d(z)$ is the coefficient appearing in the Sturm–Liouville problem. Therefore the procedure developed in ref. [5] for the automatic solution of the eigenvalue problems is applicable for the solution of the eigenvalue problem (7) or its multilayer equivalent when the coefficients $k(z)$ and $w(z)$ are chosen as stepwise functions given by equations (2).

We note that, for each value of λ_l and ν_m, there is an associated infinite set of real eigenvalues $\mu_n\equiv\mu_{nlm}$ ($n = 1, 2,\ldots,\infty$). Once the eigenvalues and the eigenfunctions are known, the solution of the problem (1) follows immediately as a special case from the general results given in refs. [3, 4]. We obtain

$$T(x,y,z,t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\exp(-\mu_{lmn}^2t)\frac{X_l(x)Y_m(x)Z_{nlm}(z)}{N_lN_mN_{nlm}}$$

$$\times\int_0^a X_l(x')\int_0^b Y_m(y')\int_0^c w(z')Z_{nlm}(z')f(x',y',z')\,dx'\,dy'\,dz' \qquad (11a)$$

where the normalization integrals are

$$N_l\equiv\int_0^a X_l^2(x)\,dx,\quad N_m=\int_0^b Y_m^2(y)\,dy,$$

$$N_{nlm}=\int_0^c w(z)Z_{nlm}^2(z)\,dz \qquad (11b)$$

we also note that

$$\int_0^a X_l(x)\,dx=\begin{cases}0 & \text{for}\quad l=1,2,\ldots,\infty\\ a & \text{for}\quad l=0\end{cases} \qquad (12a)$$

$$\int_0^b Y_m(y)\,dy=\begin{cases}0 & \text{for}\quad m=1,2,\ldots,\infty\\ b & \text{for}\quad m=0.\end{cases} \qquad (12b)$$

Depending on the functional form of the initial condition $f(z, y, z)$ some interesting conclusions can be drawn from the solution given by equations (11).

1. The initial condition depends on the z-variable only, namely

$$f(x,y,z)\equiv f(z). \qquad (13)$$

Then introducing equations (12) and (13) into the solution (11) we obtain

$$T(z,t)=\sum_{n=1}^{\infty}\exp(-\mu_n^2t)Z_n(z)\frac{\int_0^c w(z)Z_n(z)f(z)\,dz}{\int_0^c w(z)Z_n^2(z)\,dz} \qquad (14)$$

where the eigenvalues μ_n and the eigenfunctions $Z_n(z)$ are determined from equations (7) by setting $\lambda=\nu=0$. The results given by equation (14) implies that when the initial temperature varies only across the layers (i.e. in the z-direction) for the three-dimensional multilayer slab problem obtainable from equations (1), there is no temperature variation parallel to the layers (i.e. x- and y-directions).

2. The initial condition depends on the x- and y-variables only, namely

$$f(x,y,z)\equiv f(x,y). \qquad (15)$$

For this case we introduce equation (15) into equation (11a); but the general form of the solution given by equations (11) is not altered. This result implies that when the initial temperature varies only parallel to the layers (i.e. in the x- and y-directions) in the three-dimensional multilayer slab problem obtainable equations (1), there is also a temperature variation across the layers (i.e. in the z-direction).

Clearly, the conclusions drawn from the two special cases considered above are the generalization to the three-dimensional situation of the two-dimensional multi-layer slab problem considered in refs. [1, 2]. Furthermore, the sign-count algorithm described in ref. [6] can readily be used to compute the eigenvalues associated with the composite layer problem.

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On the similarity solutions to laminar natural convection boundary layers

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INTRODUCTION

DIMENSIONAL analysis shows

$$\Pi_x = Ra_x\, Pr/(1+Pr), \tag{1}$$

in the usual nomenclature, to be the natural parameter incorporating buoyancy and the weighted contributions of both inertial and viscous effects into the correlation of heat transfer in natural convection from the vertical isothermal plate. The parameter ensures the correct limits for large and small values of the Prandtl number, where $Nu_x/\Pi_x^{1/4}$ becomes constant. Accordingly, in terms of the similarity variable

$$\eta = (y/x)(\Pi_x/4)^{1/4}, \tag{2}$$

temperature distributions prove to be nearly similar for all fluids. The analysis suggests an appropriate expression for dimensionless wall friction, pointing to an approximate analogy between heat transfer and friction.

SIMILARITY VARIABLE BY DIMENSIONAL ANALYSIS

The reduction of the number of independent variables by one for a problem governed by partial differential equations requires the search for a similarity variable which appropriately combines two of the independent variables. Among the several methods available for this search, that of dimensional analysis may be extended by a judicious account of the physical similitude of governing equations. The extension employs the concept of two-length dimensional analysis [1]. This approach may give a similarity variable that carries an appropriate weight of several physical parameters in addition to the proper combination of two independent variables. We illustrate this observation by considering the classical problem of natural convection from the vertical isothermal plate in a Newtonian fluid for which the viscous dissipation may safely be neglected [2]. This problem was recently reviewed by Martin [3].

Let u and $\theta \sim T_w - T_\infty$ denote the velocity and temperature, respectively, and $\delta \sim y$ and $l \sim y$ denote the lengths characterizing flux and flow, respectively. Equating flow to flux in the balance of thermal energy gives the characteristic velocity

$$u \sim la/\delta^2, \tag{3}$$

which is not externally imposed but is determined by the problem.

Balancing buoyancy against the sum of inertial and viscous forces in the balance of momentum gives

$$\frac{f_B}{f_I + f_V} \sim \frac{g\beta\theta}{u^2/l + vu/\delta^2} = \frac{g\beta\theta l/u^2}{1 + vl/(u\delta^2)}, \tag{4}$$

or, after eliminating u by equation (3),

$$\frac{f_B}{f_I + f_V} \sim \frac{Pr}{1+Pr}\frac{g\beta\theta l^3}{av}\left(\frac{\delta}{l}\right)^4 \sim \frac{Pr}{1+Pr} Ra_x (y/x)^4. \tag{5}$$

Equation (5) suggests the similarity variable stated by

NOMENCLATURE

- a thermal diffusivity
- C constant
- g acceleration of gravity
- l reference length in x
- Nu_x local Nusselt number
- Pr Prandtl number, v/a
- Ra_x local Rayleigh number, $g\beta(T_w - T_\infty)x^3/(av)$
- $T_w - T_\infty$ temperature difference between plate and free stream
- u velocity component in x
- x coordinate from leading edge
- y coordinate normal to plate.

Greek symbols
- β coefficient of thermal expansion
- δ reference length in y
- η similarity variable, equation (2)
- ζ dimensionless streamfunction, equation (6)
- θ dimensionless temperature, equation (6)
- v kinematic viscosity
- Π_x dimensionless number, equation (1)
- ρ density
- τ_w wall shear stress.